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On Non-Fourier Temperature Wave and Thermal Relaxation Time[1]

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In this paper, the non-Fourier effects in a material under heating flux with an actual pulse and periodic temporal profile are investigated after introducing the physical mechanism of the thermal relaxation model. By using the analytical solution of the non-Fourier hyperbolic conduction equation, a discussion about the wave characteristics of non-Fourier conduction is given, and the manner in which relaxation time affects the temperature behavior is discussed. Then a measuring method for the relaxation time is suggested for these two kinds of heating flux.

KEY WORDS: laser-pulse heating; non-Fourier heat conduction, periodic heating; relaxation time; temperature wave.

1. INTRODUCTION

The use of heat sources, such as laser and microwave, with extremely short durations or very high frequencies, has found numerous applications related to surface annealing of metals [1], sintering of ceramics [2], exhibiting microscopic heat transport dynamics [3], measuring physical properties of thin films [4], etc. In such situations, the classical Fourier heat conduction model could become inaccurate.

The Fourier model is an approximate description of the real physical process, while it is suitable for common engineering problems under most conditions. A more generalized heat conduction theory, the non-Fourier theory, was formulated about 40 years ago, in which the Fourier equation was modified to include effects of the thermal relaxation process. This was

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confirmed by some experiments exhibiting second sound in solid He several years later [5–7].

This paper presents an investigation of performing the non-Fourier behavior in materials subjected to a heating flux with an actual pulse and periodic temporal profile. First, the physical mechanism of the relaxation model is introduced. Then a discussion of the wave characteristics of non-Fourier conditions is given, and the manner in which relaxation time affects the temperature wave behavior is discussed. Finally, the measuring method for the relaxation time is suggested for these two kinds of heat flux.

2. THERMAL RELAXATION MODEL OF MATERIALS

The Fourier diffusion model predicts theoretically that a thermal disturbance at any point in a medium is instantaneously felt at every other point in the medium. Such a state of affairs is clearly not physical. To eliminate this dilemma, Cattaneo [8] and Vernotte [9] independently postulated a time-dependent relaxation model for the heat flux in solids expressed as

$$\mathbf{q}(\mathbf{r}, t) = -\lambda \nabla T - \tau_0 \frac{\partial \mathbf{q}(\mathbf{r}, t)}{\partial t} \tag{1}$$

where τ_0 is a relaxation time, $\mathbf{q}$ is the heat flux vector, $\mathbf{r}$ is the spatial vector, t is the time, and λ is the thermal conductivity. The physical significance of the relaxation behavior and the estimation method of the relaxation time have attracted significant attention. About the relaxation behavior, it could be simply concluded that there is a finite buildup time for the onset of a thermal current after a temperature gradient is imposed on a specimen. In other words, heat flow does not start instantaneously but rather grows gradually with a relaxation time. The microscopic mechanism is quite complex and dependent on the kind of material. Simply speaking, the relaxation time is associated with the communication time between microparticles such as photons, electrons, and phonons. Combining the energy conservation equation

$$-\nabla \cdot \mathbf{q} + S(\mathbf{r}, t) = c \frac{\partial T}{\partial t} \tag{2}$$

with Eq. (1) leads to a description of an unsteady temperature profile in the form of the hyperbolic equation

$$\nabla^2 T(\mathbf{r}, t) + \frac{1}{\lambda}\left[S(\mathbf{r}, t) + \tau_0 \frac{\partial S(\mathbf{r}, t)}{\partial t} \right] = \frac{1}{a} \frac{\partial T(\mathbf{r}, t)}{\partial t} + \frac{\tau_0}{a} \frac{\partial^2 T(\mathbf{r}, t)}{\partial t^2} \tag{3}$$

where $S(\mathbf{r}, t)$ is a heat source in the material, T is the temperature, c is the heat capacity per unit volume, and a is the thermal diffusivity.

Physically, the relaxation time is regarded as the macroscopic parameter of a series of microscopic interaction in a material, it describes the mean time effect of all these microscopic relaxation processes, such as electron–electron, electron–phonon, and phonon–phonon scattering. Obviously, it is not the simple summation of the characteristic time of these microscopic relaxation processes. How to relate the relaxation time to these microscopic processes is still a problem now. In other words, the estimation method of the value of the relaxation time is not established yet. All of the published works are only about the estimation method of the characteristic time for one of the single microscopic relaxation processes.

In 1963, Chester [10] advanced a method for estimating the relaxation time of a phonon gas. He assumed that the square of the thermal wave velocity was one-third of the square of the phonon group velocity s^2, then he obtained

$$\tau_{\mathrm{phonon}} = \frac{3\lambda}{s^2 c} \tag{4}$$

In 1969, Maurer [11] derived a time-dependent relaxation model for the heat flux in metals from the quantum mechanical form of the Boltzmann transport equation. In this work, the phonons were assumed to be in thermal equilibrium at all times, and the Lorentz approximation was used to treat electron–phonon interactions. He obtained the following approximate expression for the relaxation time of the electron gas

$$\tau_{\mathrm{electron}} = \frac{3m\lambda}{\pi^2 n k^2 T} \tag{5}$$

where m is the electron effective mass, n is the number of electrons per unit volume, and k is the Boltzmann constant. These two formulas are employed frequently to estimate the relaxation time of solids by researchers without experimental confirmation.

3. TEMPERATURE WAVES

In this study, the conduction heat transfer in a one-dimensional medium is considered. For heating flux, two typical temporal profiles, an actual pulse and a periodic type, are chosen because they are the most frequently used heating types in engineering. For convenience in the subsequent analysis, the real coordinate is replaced by the corresponding dimensionless one as introduced in the following:

$$\eta = \frac{x}{2\sqrt{a\tau_0}}, \qquad \xi = \frac{t}{2\tau_0} \tag{6}$$

The non-Fourier temperature wave behaviors for these two considered conditions are performed in the following two subsections.

3.1. A Finite Medium Under Laser-Pulse Heating

First, an one-dimensional heat conduction problem in a material of thickness L with initial temperature distribution $T(x, 0) = 0$ (x is the direction along the thickness), constant thermal properties, and insulated boundaries is considered. From time $t = 0$, its front surface ($x = 0$) is exposed to a laser-pulse heat flux, whose temporal shape has a typical form

$$q(t) = Q_0 \frac{t}{t_p^2} e^{-t/t_p} \tag{7}$$

where t_p and Q_0 denote the time duration and total energy of a single pulse at the unit surface, respectively. The pulsed energy is assumed to be absorbed uniformly in a small depth δ near the front surface, which can be treated as a pulse energy source in the material as follows:

$$Q(x, t) = \begin{cases} \dfrac{Q_0}{\delta}\left(\dfrac{t}{t_p^2}\right) e^{-t/t_p} & 0 \leqslant x \leqslant \delta \\ 0 & \delta < x \leqslant L \end{cases} \tag{8}$$

Then the governing equation is Eq. (3) in one-dimensional form, and the initial and boundary conditions are

$$\frac{\partial T(0, t)}{\partial x} = \frac{\partial T(L, t)}{\partial x} = 0 \tag{9}$$

$$\frac{\partial T(x, 0)}{\partial t} = 0, \qquad T(x, 0) = 0 \tag{10}$$

The analytical solution can be derived by the Green's function method and infinite integral transform technique; details are given in the literature [12].

Using the introduced dimensionless quantities of Eq. (6) and the quantities for this problem as

$$\theta(\eta, \xi) = \frac{\lambda\sqrt{\tau_0}}{Q_0\sqrt{a}}\, T(x, t), \qquad \eta_1 = \frac{L}{2\sqrt{a\tau_0}}, \qquad \Delta\eta = \frac{\delta}{2\sqrt{a\tau_0}}, \qquad \xi_p = \frac{t_p}{2\tau_0} \tag{11}$$

the solution is obtained as

$$\begin{aligned}\theta(\eta, \xi) = {} & \frac{1}{2\eta_1}\left(1 - \frac{\xi + \xi_p}{\xi_p}\, e^{-\xi/\xi_p}\right) \\ & + \frac{e^{-\xi}}{\eta_1} \sum_{n=1}^{\infty} \cos\frac{n\pi\eta}{\eta_1}\, \frac{\sin(n\pi\,\Delta\eta/\eta_1)}{n\pi\,\Delta\eta/\eta_1} \\ & \times \frac{1}{\gamma}\left\{\left[\xi(2 - 1/\xi_p) - \frac{\gamma_1}{\gamma}\right] e^{\xi(1-1/\xi_p)} + \frac{\gamma_2}{\gamma}\frac{\sin\beta\xi}{\beta} + \frac{\gamma_1}{\gamma}\cos\beta\xi\right\}\end{aligned} \tag{12}$$

where

$$\gamma = \left(\frac{n\pi}{\eta_1}\right)^2 \xi_p^2 - 2\xi_p + 1, \; \gamma_1 = \left[4 - \left(\frac{n\pi}{\eta_1}\right)^2\right] \xi_p^2 - 4\xi_p + 1 \tag{13a}$$

$$\gamma_2 = \left[4 - 3\left(\frac{n\pi}{\eta_1}\right)^2\right] \xi_p^2 - 2\left[2 - \left(\frac{n\pi}{\eta_1}\right)^2\right] \xi_p + 1 \tag{13b}$$

$$\beta = \sqrt{(n\pi/\eta_1)^2 - 1} \tag{13c}$$

Utilizing Eq. (12), numerical computations are performed to display the behavior of the temperature waves in a finite medium irradiated by a laser pulse in the form of Eq. (7), the calculated temperature waves are illustrated in Figs. 1 and 2. For the subsequent discussion to have a clear physical picture, the significance of the introduced dimensionless quantities must be well understood. From Eqs. (6) and (11),

$$\eta = \frac{x}{2\sqrt{a\tau_0}} = \frac{x}{2v\tau_0}, \qquad \text{and} \qquad \eta_1 = \frac{L}{2v\tau_0} \tag{14}$$

where $v = \sqrt{a/\tau_0}$ is the propagation speed of the temperature wave. Then η denotes the ratio of the characteristic physical length scale to the thermal length scale (mean free path, $v\tau_0$). $\xi = t/2\tau_0$ is the ratio of the physical time scale of the thermal process to the thermal relaxation time.

Figure 1 shows the temperature profiles at various positions throughout a finite medium with a thickness of $\eta_1 = 1$ irradiated by a laser pulse of

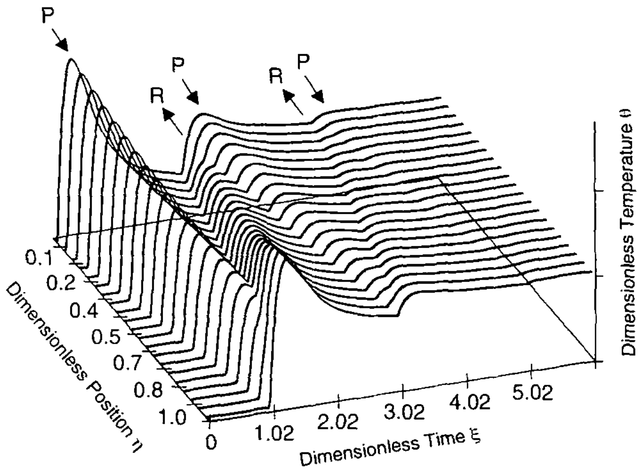

Fig. 1. Evolution of temperature profiles inside a finite medium with thickness $\eta_1 = 1$ irradiated by a laser pulse with duration $\xi_1 = 0.2$. The absorptive depth $\Delta\eta = 0.02$.

$\xi_1 = 0.2$. From the figure, the behavior of the non-Fourier temperature wave can be observed clearly. Several series of wave peaks (P and R) shown in the figure indicate the propagation and reflection of the temperature wave. It can be seen that by several times of propagation and reflection between the two surfaces of the medium, the portion of the temperature wave is dissipated and the temperature of the medium becomes uniform.

Figure 2 shows the propagation ($\xi = 0.2$ and $\xi = 0.85$) and reflection ($\xi = 1.65$) of the temperature wave inside a finite medium with a thickness

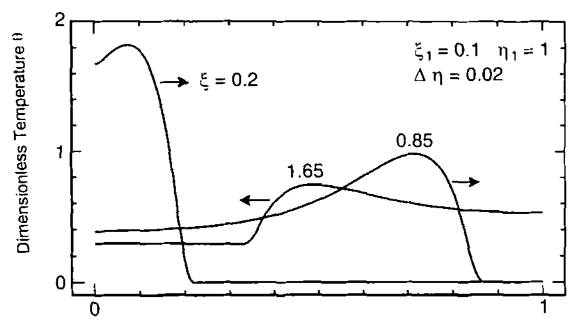

Fig. 2. Propagation of the temperature waves in a finite medium.

of $\eta_1 = 1$ irradiated by a laser with a pulse duration of $\xi_1 = 0.1$. By checking the position of the wave front at a different time, it is found that $\xi = \eta_{wf}$, where η_{wf} is the propagating distance of wave front at time ξ in a dimensionless system, for example, when $\xi = 0.2$, the wave front is at the position $\eta_{wf} = 0.2$. If it is converted to a real coordinate system by using Eqs. (6), (11), and (14) and $v = x_{wf}/t$, where x_{wf} is the propagating distance of the wave front at time t. This means that the hyperbolic heat conduction equation predicts a wave with a propagation speed of v, which is determined by the relaxation time.

3.2. A Semiinfinite Medium Under Periodic Heating

In the case of a periodic heating flux, for a semiinfinite medium the temperature wave solution was derived by Yuen and Lee [13] in 1989, and for a finite medium two forms of solutions were given recently by Tang and Araki [14, 15]. Comparison of the two results shows that the solution of a semiinfinite medium takes a relatively simple form, and its temperature response profiles are smooth periodic curves because, in this situation there is no reflection of temperature wave like that in the finite medium. For convenience, in discussion and suggestion of measuring methods of relaxation time, here we choose the semiinfinite medium. For a one-dimensional semiinfinite material subjected to a periodic surface heat flux, the governing equation is the one-dimensional form of Eq. (3) with heat source $S(\mathbf{r}, t) = 0$; the boundary and initial conditions are given by

$$-\lambda \frac{\partial T(0, t)}{\partial x} = q_0 e^{i\omega t} \tag{15a}$$

$$T(\infty, t) = \frac{\partial T(\infty, t)}{\partial x} = 0 \tag{15b}$$

$$T(x, 0) = \frac{\partial T(x, 0)}{\partial x} = 0 \tag{15c}$$

where q_0, and ω are the amplitude and frequency of the surface heat flux oscillation, respectively. Using the introduced dimensionless quantities of Eq. (6) and the quantities for this problem

$$\theta = \frac{\lambda}{q_0 \sqrt{a\tau_0}} T(x, t), \qquad \xi_\omega = \frac{1}{2\omega\tau_0} \tag{16}$$

the temperature response is obtained as

$$\theta(\eta, \xi) = H(\xi - \eta)\, I_0(\sqrt{\xi^2 - \eta^2}) + \left(2 + \frac{i}{\xi_\omega}\right) e^{i(\xi/\xi_\omega)} \int_\eta^\xi e^{-i(\xi'/\xi_\omega) - \xi'} I_0(\sqrt{\xi'^2 - \eta^2})\, d\xi' \tag{17}$$

where $H(\ )$ is the Heaviside unit step function and $I_0(\ )$ is the modified Bessel function of zero order. For a periodic temperature response, the most useful form is its long time limit in actual applications. Let $\xi \to \infty$ (i.e., $t \to \infty$); the temperature profile can be written in closed form as

$$\theta(\eta, \xi) = \frac{[1 + (1/2\xi_\omega)^2]^{1/4}}{\sqrt{1/2\xi_\omega}}\, e^{-(K_+ - K_-)\eta}\, e^{i((\xi/\xi_\omega) + r) - i(K_+ + K_-)\eta} \tag{18}$$

where

$$K_+ = \left[\frac{1}{2\xi_\omega}\left(\sqrt{1 + \left(\frac{1}{2\xi_\omega}\right)^2} + 1\right)\right]^{1/2}, \quad K_- = \left[\frac{1}{2\xi_\omega}\left(\sqrt{1 + \left(\frac{1}{2\xi_\omega}\right)^2} - 1\right)\right]^{1/2} \tag{19a}$$

$$r = \frac{1}{2}\tan^{-1}\left(\frac{1}{2\xi_\omega}\right) - \frac{\pi}{4} \tag{19b}$$

Utilizing Eq. (18), numerical computations are performed to display the behavior of the temperature waves in a semiinfinite medium irradiated by a periodic heat flux in the form of Eq. (15a); the calculated temperature waves are illustrated in Fig. 3. In Fig. 3, the temperature profiles at various positions from $\eta = 0$ to $\eta = 1.5$ in the semiinfinite medium subjected to a periodic heat flux with period of $\xi_\omega = 0.5$ are shown. Here the starting point of time ($\xi = 0$) is actually a moment after a relative long time from the beginning of heating. Because there is no reflection of temperature wave for a semiinfinite medium, the temperature response curves have the same form as that from Fourier theory, without a jump point on them as in a finite medium. The differences from those of the Fourier model are that they have different phase lag and amplitude differences after propagation of a distance. The non-Fourier phase lag and amplitude difference are expressed as

$$\frac{\Delta\varphi_H}{\Delta\varphi_P} = K_+ + K_- \tag{20a}$$

$$\frac{\Delta A_H}{\Delta A_P} = K_+ - K_- \tag{20b}$$

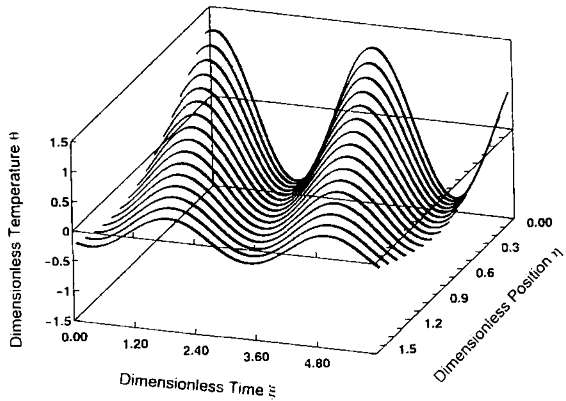

Fig. 3. Evolution of temperature profiles inside a semiinfinite medium exposed to a periodic heating flux with frequency $\xi_\omega = 0.5$ on the surface ($\eta = 0$).

where $\Delta\varphi(=\varphi_2 - \varphi_1)$ and $\Delta A(=A_1/A_2)$ are the phase lag and amplitude ratio of temperature responses at two different positions in the medium, and the subscripts H and P denote non-Fourier and Fourier. From Eq. (19), with $\tau_0 \to 0$, Eq. (20) approaches 1 and the non-Fourier result reduces to the Fourier one.

4. MEASUREMENT METHOD OF RELAXATION TIME

According to the discussion about the two kinds of heating flux, the non-Fourier behavior of heat conduction is determined mainly by the relaxation time, that is, the non-Fourier behavior of temperature response should reflect the thermal relaxation processes. Then inversely, it is possible for us to obtain the relaxation time by using the measured temperature responses. In the following, the measurement method for the two considered kinds of heating flux is suggested.

4.1. For a Pulse Heating Flux

Actually the temperature wave always exists inside the film irradiated by a laser pulse. But at the rear surface the wave characteristics cannot always appear, or the wave front cannot always be distinguished. The possibility of observation of the temperature wave is determined mainly by three factors: the thickness of the film, heat pulse duration, and absorption depth. The calculation in the preceding section shows that, when the absorption depth is small enough, for $\eta_l \sim 1$ and $\xi_l \sim 0.5$, i.e., $L \sim 2\sqrt{a\tau_0}$ and $t_p \sim \tau_0$, the temperature wave is surely able to be felt at the rear surface. As an example, a metal with high thermal diffusivity is considered, at room temperature, $a = 1.2 \times 10^{-4}\,\mathrm{m}^2 \cdot \mathrm{s}^{-1}$, and $\tau_0 = 10^{-9}$–10^{-12} s [16], the experimental condition can be estimated as $t_p = 10^{-9}$–10^{-12} s and $L = 6.9 \times 10^{-7}$–2.2×10^{-8} m. According to the above calculation results, it can be seen that, $\Delta\xi = 2\eta_l$, where $\Delta\xi$ is the time difference between the two peaks in the curve of dimensionless temperature response at the rear surface, then along with Eq. (11), the following is obtained:

$$\frac{\Delta t}{2\tau_0} = \frac{2L}{2\sqrt{a\tau_0}}, \qquad \text{i.e.,} \quad \tau_0 = \frac{a\,\Delta t^2}{4L^2} \tag{21}$$

where Δt is the real time difference between the two peaks in the temperature response curve. If the temperature response at the rear surface is observed, Δt would be measured. Then the relaxation time can be calculated from Eq. (21).

4.2. For a Periodic Heating Flux

According to Eq. (16) and Eq. (18), the phase lag and amplitude ratio of the non-Fourier temperature responses at two positions with Δx apart can be expressed as follows:

$$\Delta\varphi_H = \left[\sqrt{\frac{\sqrt{1+(\omega\tau_0)^2}+1}{2}} + \sqrt{\frac{\sqrt{1+(\omega\tau_0)^2}-1}{2}}\right]\sqrt{\frac{\omega}{2a}}\,\Delta x \tag{22}$$

$$\Delta A_H = \left[\sqrt{\frac{\sqrt{1+(\omega\tau_0)^2}+1}{2}} - \sqrt{\frac{\sqrt{1+(\omega\tau_0)^2}-1}{2}}\right]\sqrt{\frac{\omega}{2a}}\,\Delta x \tag{23}$$

Rewriting Eqs. (22) and (23), we obtain the expression of the relaxation time as

$$\tau_0 = \frac{B^2-1}{2B\omega}, \qquad \text{with} \quad B = \frac{2a\,\Delta\varphi_H^2}{\omega\,\Delta x^2} \tag{24}$$

$$\tau_0 = \frac{1-C^2}{2C\omega}, \qquad \text{with} \quad C = \frac{2a\,\Delta A_H^2}{\omega\,\Delta x^2} \tag{25}$$

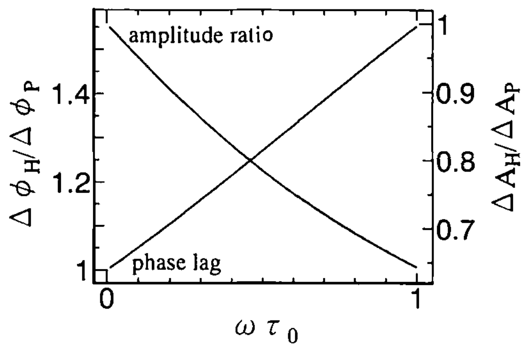

Fig. 4. Amplitude ratio and phase lag ratio of the non-Fourier to the Fourier conduction as functions of $\omega\tau_0$.

Then if the phase lag or amplitude ratio is measured, by using Eq. (24) or (25), the relaxation time can be obtained. For the Fourier heat conduction, according to the formula of the phase lag and amplitude ratio, we have $B = 1$ and $C = 1$, then $\tau_0 = 0$. From Eq. (20), the amplitude ratio and the phase lag ratio of the non-Fourier to the Fourier conduction are functions of $\omega\tau_0$ (as shown in Fig. 4). Figure 4 shows that the Fourier heat conduction predicts the minimum limit of the phase lag and the maximum damping of the amplitude ratio. From the figure, when $\omega\tau_0 > 1$, i.e., $\omega > 1/\tau_0$, the phase lag ratio of the non-Fourier to the Fourier is bigger than 1.6, the non-Fourier effect is still quite evident. Considering the same materials as discussed in the above section for pulse heating, if we want to observe the non-Fourier effect, the heating flux should have a frequency of $\omega > 10^9 - 10^{12}$ Hz. From Eq. (18), it is known that temperature responses would also change with this frequency. When we choose two measurement points in the material which are 1 cm apart, i.e., $\Delta x = 1$ cm, according to Eq. (23), the amplitude decay should be $\Delta A_H \approx 10^4$–10^5, which means that the amplitude at the second measurement point would be 10^4–10^5 times smaller than that of the first point. Therefore, we have to measure and compare two periodic temperature responses whose amplitudes are quite different from each other.

5. CONCLUDING REMARKS

By using analytical solutions of the non-Fourier hyperbolic conduction equation, the wave behavior of heat conduction in a finite and a semiinfinite medium under an actual pulse and a periodic heating flux is demonstrated. The effect of the relaxation time on the temperature response is investigated and the interrelation between some measurable quantities from the temperature responses and the relaxation time is established by Eq. (21) for pulse heating and Eqs. (24) and (25) for periodic heating. This makes it possible for us to suggest the measurement method and conditions for the relaxation time. The reliability of these methods must be checked by experiments, which will be undertaken in the future.

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